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On Dirac's conjecture for systems having only first-class constraints

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Abstract. Dirac's conjecture is proved for a general class of systems having only first-class constraints. In other words it is shown for that kind of problem that all the primary or secondary first-class constraints generate equivalence transformations between physical states.

1. Introduction

Dirac's generalised canonical formalism (Dirac 1950, 1964) plays at present a relevant role in modern quantum field theory. By using it, many of the central problems which have appeared in the development of the quantisation procedures of the gauge and gravitational fields have been solved (Faddeev 1970, Fradkin and Vilkovisky 1975). The method has been further perfected to a great extent to make it applicable not only to real and complex fields (bosonic fields) but also to fields which take values in Grassmann algebras (fermionic fields) (Casalbuoni 1976).

At the same time, by starting from this procedure, path integral quantisation schemes for systems having arbitrary numbers of first- and second-class constraints have been created (Batalin and Vilkovisky 1977, Fradkin and Fradkina 1978).

However, in spite of these general achievements some basic problems in this theory are still widely discussed in the literature. One of them is related to the equivalence between Dirac's procedure in terms of the extended Hamiltonian and the Lagrangian description (Cawley 1979, Frenkel 1980). This problem in its turn is closely connected with the question about whether all the first-class constraints (primary or secondary) are generators of gauge transformations (Dirac 1964, Gitman and Tyutin 1983, Di Stefano 1983, Sugano and Kamo 1982, Sugano 1982, Sugano and Kimura 1983).

The present work is linked with the second of these controversial aspects. It is shown for a wide class of constrained dynamical systems in which there are only first-class constraints that all of these generate infinitesimal canonical transformations forming physically equivalent states. The meaning of equivalence accepted by us is the same as the one used by Dirac (1964): two points in the phase space (states) are considered as equivalent when they evolve from another point in a previous instant of time according to two total Hamiltonians coming from the same Lagrange system, i.e. they differ at most in the Lagrange multipliers of the primary first-class constraints. In obtaining this result we have appealed to some basic notions appearing in the papers of Sugano (1982) and Sugano and Kamo (1982).

In § 2 we prove the possibility of giving a particular structure to a complete set of first-class constraints. This result is obtained for a wide class of systems which are specialised there. All the other results of the remaining sections concern this kind of system.

In § 3 a sufficient condition is developed for a linear combination of primary or secondary first-class constraints ψ to generate an infinitesimal canonical transformation mapping an extremal of the canonical action into another one. It follows that for the systems defined in § 2 there always exists some ψ satisfying this condition.

Section 4 contains the proof of Dirac's conjecture, i.e. the fact that all the first-class constraints (primary or secondary) generate equivalence transformations among physical states.

Finally in § 5 an example is examined in detail in order to illustrate the main points exposed in the work.

2. First-class constraint ordering

We begin by presenting some basic definitions in Dirac's method. We start from the action

$$S=\int_{t_1}^{t_2} L(q, \dot{q})\, \mathrm{d}t \tag{1}$$

where L is supposed to be a function of the coordinates q_i, $i=1, \ldots, N$, and their time derivatives $\dot{q}_i$. The Lagrangian L is called singular if the Hessian matrix

$$H_{ij}=\partial^2 L(q, \dot{q})/\partial \dot{q}_i\, \partial \dot{q}_j$$

has its rank M less than the number of degrees of freedom N. Then it is not possible to eliminate all the velocities $\dot{q}_i$ as functions of the momenta p_i by using solely their definitions

$$p_i=\partial L(q, \dot{q})/\partial \dot{q}_i, \qquad i=1, \ldots, N. \tag{2}$$

This situation implies the existence of $m=N-M$ independent relations between the q_i and p_i (primary constraints) of the form

$$\varphi^a(q, p)=0, \qquad a=1, \ldots, m. \tag{3}$$

It will be supposed in what follows that the set of φ^a is complete in the sense of Dirac (1950): a system of constraint functions is complete when any function vanishing on the manifold spanned by all the points satisfying the set of constraints can be expressed as a linear combination (with (q, p)-dependent coefficients) of the constraint functions.

Also, it is convenient to make precise our conception about a set of independent constraints. A collection of S constraint functions or conditions is independent if the set of points which satisfy simultaneously all the constraints is a manifold of dimension $2N-S$ on the phase space. Henceforth the name 'constraint' will be conventionally used both for the constraint $\chi=0$ itself and for the function χ.

Under the assumption of the completeness of the set of primary constraints, it can be proved that the Lagrange equations of motion are equivalent to the canonical equations following from the total Hamiltonian

$$H_{\mathrm{T}}=H(q, p)+\lambda^a \varphi^a(q, p) \tag{4}$$

where λ^a are the Lagrange multipliers to the primary constraints (3) (the summation over repeated indices is accepted throughout the paper). The Hamiltonian $H(q, p)$ in (4) is defined as follows:

$$H(q, p) = p_i \dot{q}_i - L(q, \dot{q}) \tag{5}$$

on the manifold P which is determined by all the primary constraints. Outside P the function H is defined in an arbitrary but analytical way.

We concentrate now on proving the existence of a special type of complete set of constraints in the class of systems to be characterised below.

Let a system be described by the total Hamiltonian (4) and suppose that Dirac's procedure for the determination of the multipliers and constraints is already performed. Suppose also that the resulting set

$$FC = \{\varphi_a, a = 1, \dots, m; \psi_k, k = 1, \dots, f\}$$

of all constraints contains only first-class ones and is complete in the above-mentioned sense. Then, the class of constrained systems to which the results presented here refer is fully specified by the condition that all the constraints arising in Dirac's procedure have been obtained only by forming linear combinations (with bounded (q, p)-dependent coefficients) of the consistency conditions.

Under this definition, it follows that any of the first-class constraint functions of the problem can be expressed as a linear combination of the primary constraints and the repeated Poisson brackets of these with the Hamiltonian H. By a repeated Poisson bracket of $A(q, p)$ with H we mean the function

$$B(q, p) = \hat{X}^n A(q, p) \tag{6}$$

for some non-negative integer n. In (6) the linear operator $\hat{X}$ acting on a function $f(q, p)$ is given by the Poisson bracket of f with H:

$$\hat{X} f(q, p) = \{f, H\}. \tag{7}$$

Moreover, the first-class nature of all the constraints of the system implies that H and φ^a are first-class functions. Then it can be concluded also that all the repeated commutators of the φ^a with H vanish in the manifold PG which is defined as the set of points satisfying all the constraints from FC.

From this fact and the completeness of the set FC it follows that any repeated Poisson bracket of the φ^a with H is a linear combination of all the constraints defining FC.

Consider now the collection of constraint functions constructed as follows:

$$\begin{aligned}
\psi_0^a &= \varphi^a \\
\psi_1^a &= \hat{X}\varphi^a \\
&\vdots \\
\psi_n^a &= \hat{X}^n \varphi^a \\
&\vdots \\
\psi_{r(a)}^a &= \hat{X}^{r(a)} \varphi^a
\end{aligned} \tag{8}$$

for each $a = 1, \dots, m$.

It is clear from the above considerations that for some set of integers $r(a)$, $a=1,\ldots,m$, the following relations must hold:

$$\{\psi^a_{r(a)}, H\} = \hat{X}^{r(a)+1}\varphi^a = \sum_{a'=1}^{m}\sum_{n=0}^{r(a')} R^{aa'}_n(q,p)\psi^{a'}_n(q,p), \qquad a=1,\ldots,m. \tag{9}$$

After this, by selecting conveniently the numbers $r(a)$, it is always possible to include in the set of the ψ^a_n all the repeated Poisson brackets of the φ^a with H through which any of the first-class constraints of the system is expressed as a linear combination. The set of the functions ψ^a_n is complete but not necessarily independent.

3. Generators which map solutions into solutions

Let us show that in the class of problems defined in § 2 there exists an infinitesimal linear combination of the first-class constraints $\varepsilon\psi(q,p)$ which generates a canonical transformation mapping an extremum of the action into another one; ε is a constant infinitesimal parameter and ψ does not depend explicitly on time.

From the first-class nature of ψ it follows directly that the manifold *PG* remains invariant under the canonical transformation induced by $\varepsilon\psi$.

Due to the fact that in the kind of systems considered here there are only first-class constraints, the Lagrange multipliers are in no way determined. But, from the very beginning the Hamiltonian $H(q,p)$ is only defined in the manifold P traced out by the primary first-class constraints φ^a. Consequently, outside P there remains for H an arbitrariness in any linear combination of the φ^a. If changed within the range of this arbitrariness, H may be still considered as the total Hamiltonian of the system. This fact will be used in the following discussion.

Let $K(q,p)$ be the Hamiltonian which is obtained from $H(q,p)$ by the transformation produced by the generator $\varepsilon\psi$ in some open neighbourhood *EA* of a trajectory. The coincidence of K and H will be required in *all EA* modulo a linear combination of the first-class primary constraints φ^a. This condition ensures that the transformed trajectory corresponds also to an extremal. This is so, because the Hamiltonian changes only in a linear combination of the primary constraints and also all these continue to be satisfied if they were before.

The new Hamiltonian K, after the transformation generated by $\varepsilon\psi$, is given by (Goldstein 1950)

$$K(Q,P) = H(q,p) + \varepsilon\frac{\partial\psi}{\partial t}(q,p) = H(q,p) \tag{10}$$

where the new variables (Q,P) are related to the old ones by

$$Q_i = q_i + \varepsilon\{q_i, \psi\}, \qquad P_i = p_i + \varepsilon\{p_i, \psi\}. \tag{11}$$

Using (11), equation (10), within the approximation linear in ε, becomes

$$K(q,p) = H(q,p) + \varepsilon\{\psi, H\}. \tag{12}$$

In (10) and (12) the time derivative of ψ does not appear because ψ was supposed not to depend explicitly on time. This assumption simplifies the discussion although it leaves sufficient generality for our needs.

From (12) we see that the above-mentioned condition on ψ can be written as

$$\{\psi, H\} = \omega_a(q,p)\varphi^a \tag{13}$$

in the open neighbourhood EA of the trajectory. The fulfilment of (13) ensures that the transformed trajectories obey the same Hamiltonian equations coming from H. The above conclusion is valid because the only change introduced in the equations of motion consists in a modification of the arbitrary multipliers associated to the primary first-class constraints. It may also be argued that the existence of ψ satisfying (13) implies that the transformation induced by it can be integrated out to a finite canonical mapping. The condition (13) was introduced previously by Sugano and Kamo (1982).

Let us continue by showing the existence of the function ψ satisfying (13) for the systems defined in § 2.

Let ψ be expressed as a linear combination of the constraints ψ_n^a given in (8) as follows:

$$\psi(q,p)=\sum_{a=1}^{m}\sum_{n=0}^{r(a)} C_n^a(q,p)\psi_n^a(q,p). \tag{14}$$

By substituting (14) for ψ in (13) one obtains

$$\sum_{a=1}^{m}\sum_{n=0}^{r(a)} (C_n^a\{\psi_n^a, H\}+\{C_n^a, H\}\psi_n^a)=\sum_{a=1}^{m}\omega_a\varphi^a. \tag{15}$$

Now, by using (8) and (9) in (15) one obtains

$$\sum_{a=1}^{m}\left(\{C_0^a, H\}+\sum_{a'=1}^{m} C_{r(a')}^{a'}R_0^{a'a}\right)\varphi^a$$
$$+\sum_{a=1}^{m}\sum_{n=1}^{r(a)}\left(\{C_n^a, H\}+C_{n-1}^a+\sum_{a'=1}^{m} C_{r(a')}^{a'}R_n^{a'a}\right)\psi_n^a=\sum_{a=1}^{m}\omega_a\varphi^a. \tag{16}$$

By requiring that in EA each coefficient of the secondary constraints in (16) vanishes, the following equations for $C_n^a(q,p)$ arise:

$$\{C_n^a, H\}+C_{n-1}^a+\sum_{a'=1}^{m} C_{r(a')}^{a'}R_n^{a'a}=0, \qquad n=1,\ldots,r(a), \qquad a=1,\ldots,m, \tag{17}$$

while the vanishing of the coefficients of the primary constraints φ^a results in the relation determining the functions ω^a,

$$\{C_0^a, H\}+\sum_{a'=1}^{m} C_{r(a')}^{a'}R_0^{a'a}=\omega^a.$$

Suppose now that in (14) all the quantities $C_{r(a)}^a$ are given in an arbitrary way as functions of (q,p). Then, for each a in (17), the coefficient C_{n-1}^a can be found if C_n^a is known. Hence, every C_n^a (for any fixed a) can be found by using successively equations (17) for the fixed value of a and decreasing values of n starting from $n=r(a)$.

In this way the desired result arises: the transformation ψ obeying (13) always exists for the class of systems under consideration and may be characterised by the set of arbitrary functions $C_{r(a)}^a$ of p and q, whose number is equal to that of the primary first-class constraints m. The latter statement will prove to be in agreement with Gitman and Tyutin (1985) after we show in § 4 that ψ performs an equivalence (gauge) transformation.

4. Dirac's conjecture

Now that all the basic preliminaries have been given, it will be shown in this section that the canonical transformations which are generated by any linear combination of first-class constraints map points in the phase space into physically equivalent ones.

By physical equivalence of the points (q, p) and (Q, P) in the phase space we understand (following Dirac (1964)) that if the pair (q, p) belongs to some solution C of the equation of motion, then the pair (Q, P) belongs to another solution C', where C and C' satisfy the same initial conditions.

Represent equation (17) as follows:

$$\begin{aligned}
C^a_{r(a)-1} &= -\hat{X}C^a_{r(a)} + P_1(C^a_{r(a)}) \\
&\vdots \\
C^a_{r(a)-n} &= (-1)^n\hat{X}^n C^a_{r(a)} + P_2(C^a_{r(a)}, \hat{X}C^a_{r(a)}, \ldots, \hat{X}^{n-1}C^a_{r(a)}) \qquad a = 1, \ldots, m, \\
&\vdots \\
C^a_0 &= (-1)^{r(a)}\hat{X}^{r(a)}C^a_{r(a)} + P_{r(a)}(C^a_{r(a)}, \hat{X}C^a_{r(a)}, \ldots, \hat{X}^{r(a)-1}C^a_{r(a)}),
\end{aligned} \tag{18}$$

where P_n are polynomial in their arguments.

Consider now the field of directions F determined by the phase space vectors

$$F = (\partial H/\partial p_i, -\partial H/\partial q_i) \tag{19}$$

and a point $\mathscr{P} = (q, p)$ through which a 'line of force' of the field F passes. This line of force coincides with the trajectory passing through $\mathscr{P}$.

Then $\hat{X}^n C^a_{r(a)}$ evaluated at the point $\mathscr{P}$ (supposed not to be a singular point of the field F) is the n-fold derivative of $C^a_{r(a)}$ along the vector F in $\mathscr{P}$, i.e. the time derivative along the trajectory.

Referring to the arbitrariness of the coefficient functions $C^a_{r(a)}(p, q)$ on the phase space left by equation (13), we can affix also arbitrary values to any number of their derivatives along F at the fixed point of the phase space $\mathscr{P}$. With these derivatives assigned, equation (18) allows us to calculate the coefficients C^a_n at $\mathscr{P}$. Vice versa, once the set of numbers $C^a_{r(a)}, C^a_{r(a)-1}, \ldots, C^a_0$, which are the values of the corresponding coefficient functions evaluated in the given point $\mathscr{P}$, is given, the directional derivatives $\hat{X}^n C^a_{r(n)}$, $n = 1, 2, \ldots, r(a)$, may be found directly from (18) by a step-by-step procedure starting with the first equation (18), $\hat{X}C^a_{r(a)} = P_1(C^a_{r(a)}) - C^a_{r(a)-1}$, etc. Thus, we see that to fix the values of C^a_n in a particular point it is sufficient to specify all the quantities $C^a_{r(a)}$ and the finite number of their time derivatives along the trajectory in the same point $\mathscr{P}$. This means that outside any open neighbourhood of $\mathscr{P}$ in the phase space all the functions $C^a_{r(a)}(p, q)$ remain completely arbitrary. In particular they may be defined as vanishing outside some open region containing the point $\mathscr{P}$. In this case outside this region the trajectory is not transformed by the generator ψ, and the latter canonically maps into one another trajectories which coincide in the past and future of the point $\mathscr{P}$ and its neighbourhood.

Consider now any linear combination θ of the first-class constraints of the system

$$\theta(q, p) = \sum_{a=1}^{m} \sum_{n=0}^{r(a)} \theta^a_n(q, p)\psi^a_n(q, p). \tag{20}$$

Let $\mathscr{P} = (p, q)$ be a point which belongs to a solution C (trajectory) of the Hamiltonian problem (a physical state) required. In (20) the quantities θ^a_n may also depend on a number of other parameters including the time.

The transformation induced by θ is given by

$$Q = q + \varepsilon \sum_{a=1}^{m} \sum_{n=0}^{r(a)} \{q, \psi_n^a(q, p)\}\theta_n^a(q, p),$$

$$P = p + \varepsilon \sum_{a=1}^{m} \sum_{n=0}^{r(a)} \{p, \psi_n^a(p, q)\}\theta_n^a(q, p). \tag{21}$$

In obtaining (21) use was made of the fact that in the space restricted by all the constraints every ψ_n^a vanishes. As was seen before, a function $\varepsilon\psi$ expanded as

$$\varepsilon\psi(q, p) = \varepsilon \sum_{a=1}^{m} \sum_{n=0}^{r(a)} C_n^a(q, p)\psi_n^a(q, p) \tag{22}$$

with the basis functions ψ_n^a defined according to (8) can always be constructed in such a way that in a special point $\mathscr{P}$ all the quantities C_n^a coincide with the given set θ_n^a. Furthermore, as was mentioned above, outside some open neighbourhood N of $\mathscr{P}$ all the C_n^a can be fixed to be zero.

Hence, by noticing that the transformation induced by $\varepsilon\psi$ at $\mathscr{P}$ coincides exactly with the one generated by θ and also that $\varepsilon\psi$ transforms extremals into extremals, we conclude that the transformed point (Q, P) belongs to another solution C'.

Finally, by remembering that outside some open neighbourhood N of $\mathscr{P}$ we can make the function $\varepsilon\psi$ vanish smoothly, it follows that C and C' may join smoothly together. Then the points (Q, P) and (q, p) are physically equivalent.

Thus, we have shown that, given any superposition of first-class constraints $\theta(p, q)$, (20), and a point $\mathscr{P}$ belonging to a trajectory C, other superpositions $\psi(p, q)$ subject to equation (13) may be found, such that $\theta(\mathscr{P}) = \psi(\mathscr{P})$, which map canonically the trajectory C into any (physically equivalent) trajectories $C'_{\mathscr{P}}$, differing from C in our open neighbourhood of $\mathscr{P}$ and coinciding with C outside this neighbourhood. Simultaneously, the point $\mathscr{P}$ is mapped by $\theta(p, q)$ into a point $\mathscr{P}'$, belonging to every $C'_{\mathscr{P}}$. This does not imply that $\theta(p, q)$ transforms a finite fragment of a trajectory into a fragment of a physically equivalent one.

5. Example

In this section we discuss an example which illustrates the main concepts appearing in this work.

The system is defined by the Lagrangian

$$L = \dot{q}_1\dot{q}_3 + \tfrac{1}{2}q_2(q_3^2 - a^2) \tag{23}$$

which gives the following canonical momenta:

$$p_1 = \partial L/\partial \dot{q}_1 = \dot{q}_3, \qquad p_2 = \partial L/\partial \dot{q}_2 = 0, \qquad p_3 = \partial L/\partial \dot{q}_3 = \dot{q}_1. \tag{24}$$

The Lagrange equations coming from (28) are

$$\ddot{q}_3 = 0, \qquad q_3^2 - a^2 = 0, \qquad \ddot{q}_1 = q_2 q_3. \tag{25}$$

From (24) it follows that there exists only one primary constraint

$$\varphi = p_2 = 0 \tag{26}$$

defining the manifold P as the plane $p_2 = 0$ and satisfying the completeness condition.

The Hamiltonian H is given by

$$H = \sum_{i=1}^{3} p_i \dot{q}_i - L = p_1 p_3 - \tfrac{1}{2} q_2 (q_3^2 - a^2) \tag{27}$$

in P. Outside P it is

$$H = p_1 p_3 - \tfrac{1}{2} q_2 (q_3^2 - a^2) + g p_2 \tag{28}$$

with $g = g(q, p)$ being arbitrary.

The total Hamiltonian is defined as

$$H_{\mathrm{T}} = H + \lambda p_2 = p_1 p_3 - \tfrac{1}{2} q_2 (q_3^2 - a^2) + g p_2 + \lambda p_2. \tag{29}$$

Applying the condition of consistency for $\varphi = p_2$ we must have in P

$$\{p_2, H_{\mathrm{T}}\} = \tfrac{1}{2}(q_3^2 - a^2) = \chi_1 \approx 0 \tag{30}$$

where $\approx$ means equality in the points of P.

From (30) it is clearly seen that χ_1 is independent of $\varphi = p_2$. Both constraints satisfy the completeness condition in the manifold P_1 which is defined as the line of intersection of the planes $p_2 = 0$ and $q_3^2 = a^2$.

Imposing again the consistency condition upon χ_1, one must have in P

$$\{\chi_1, H_{\mathrm{T}}\} = p_1 p_3 = \chi_2 \approx 0. \tag{31}$$

The function χ_2 is also a constraint independent of the previous ones. Together with them it forms a complete set in the intersection of the surfaces $p_2 = 0$, $q_3^2 = a^2$ and $p_1 = 0$.

At the next step the procedure finishes because the consistency relation for χ_2 leads to

$$\{\chi_2, H_{\mathrm{T}}\} = p_1 p_1 \approx 0 \tag{32}$$

which is automatically satisfied by virtue of (30) and (31). Thus, the procedure ends up with the resulting set of constraints

$$\varphi = p_2 = 0, \qquad \chi_1 = \tfrac{1}{2}(q_3^2 - a^2) = 0, \qquad \chi_2 = q_3 p_1 = 0, \tag{33}$$

which are all first class in PG, i.e. in the manifold determined by the constraints (33).

The equations of motion generated by the total Hamiltonian are

$$\begin{aligned} &\dot{q}_1 = p_3, \qquad &&\dot{p}_1 = 0, \qquad &&p_2 = 0, \\ &\dot{q}_2 = \lambda, &&\dot{p}_2 = \tfrac{1}{2}(q_3^2 - a^2), &&q_3^2 - a^2 = 0, \\ &\dot{q}_3 = p_1, &&\dot{p}_3 = q_2 q_3, &&q_3 p_1 = 0. \end{aligned} \tag{34}$$

From (34) the following equations for the coordinates above can be extracted:

$$\ddot{q}_1 = q_2 q_3, \qquad \ddot{q}_2 = \dot{\lambda}, \qquad \ddot{q}_3 = 0, \qquad q_3^3 - a^2 = 0. \tag{35}$$

Equations (35) are completely equivalent to the Lagrange equations (25).

Reciprocally letting q_i satisfy the Lagrange equations (35) and defining p_i by (24), one sees that the canonical equations (34) are also satisfied. Thus, the equivalence of the Lagrangian and Hamiltonian coordinates of the solutions is verified. On the other hand, the lack of equivalence among the coordinates corresponding to the Lagrangian and the extended Hamiltonian solutions (Cawley 1979) will be made explicit below. The extended Hamiltonian H_{E} may be written as

$$H_{\mathrm{E}} = H_{\mathrm{T}} + \sigma \chi_1 + \gamma \chi_2 \tag{36}$$

where σ and γ are new multipliers. From (36) we can obtain the following canonical equations:

$$
\begin{aligned}
&\ddot{q}_1 = p_3, \qquad && \dot{p}_1 = 0, \qquad && p_2 = 0,\\
&\dot{q}_2 = \lambda, && \dot{p}_2 = \tfrac{1}{2}(q_3^2 - a^2), && \tfrac{1}{2}(q_3^2 - a^2) = 0,\\
&\dot{q}_3 = p_1, && \dot{p}_3 = q_2 q_3 - \sigma q_3 - \gamma, && q_3 p_1 = 0.
\end{aligned} \tag{37}
$$

From (37) the following equations for the coordinates can be deduced:

$$
\begin{aligned}
&\ddot{q}_1 = q_2 q_3 - \sigma q_3 - \gamma, \qquad && \ddot{q}_2 = \dot{\lambda},\\
&\ddot{q}_3 = 0, && q_3^2 - a^2 = 0.
\end{aligned} \tag{38}
$$

The relations (38) show clearly the inequivalence between the coordinates of the Lagrangian and the H_{E} Hamiltonian solutions.

Let us now build the set of functions ψ_n as follows:

$$
\begin{aligned}
&\psi_0 = \varphi = p_2, \qquad \psi_1 = \{\varphi, H\} = \tfrac{1}{2}(q_3^2 - a^2),\\
&\psi_2 = \{\{\varphi, H\}, H\} = q_3 p_1,
\end{aligned} \tag{39}
$$

where g was taken as zero in (29) for simplicity. Thus, the generator ψ can be written as

$$\psi(q, p) = C_0(q, p) p_2 + C_1(q, p)(q_3^2 - a^2)/2 + C_2(q, p) q_3 p_1 \tag{40}$$

upon which the condition

$$\{\psi, H\} = \omega p_2 \tag{41}$$

is to be imposed. Substituting (40) in (41), the following equations are obtained:

$$C_1 = -\{C_2, H\}, \qquad C_0 = \{\{C_2, H\}, H\} - (p_1/q_3) C_2, \qquad \omega = 0. \tag{42}$$

They show the existence of the solutions for the generator parametrised by the arbitrary function C_2. Let us take now, for example, the secondary constraint ($\theta_n(q, p) = \delta_{n2}$) in (20)

$$\theta(q, p) = q_3 p_1 \tag{43}$$

and construct the associated equivalence transformation $\psi(q, p)$.

In the point $\mathscr{P}$ one finds from the condition $\theta(\mathscr{P}) = \psi(\mathscr{P})$ that

$$C_2(\mathscr{P}) = 1, \tag{44}$$

$$C_1(\mathscr{P}) = C_0(\mathscr{P}) = 0. \tag{45}$$

Then (42) implies that in the same point $\mathscr{P}$

$$\mathrm{d}C_2/\mathrm{d}t = 0, \qquad \mathrm{d}^2 C_2/\mathrm{d}t^2 = p_1/q_3 \tag{46}$$

where the quantities $\mathrm{d}C_2/\mathrm{d}t$ and $\mathrm{d}^2C_2/\mathrm{d}t^2$ are the values of the derivatives of the function $C_2(q(t), p(t))$ considered as a function of time along the solution C evaluated at the time $t_{\mathscr{P}}$ corresponding to the point $\mathscr{P}$. Equations (44)–(46) are the only restrictions imposed on the function $C_2(q, p)$ by equation (41). They leave enough freedom for choosing the function C_2 vanishing identically outside any open time interval containing $t_{\mathscr{P}}$. The transformation generated by $\varepsilon\psi$ (after fixing the function $C_2(q, p)$ consistently with its values $C_2(t)$ along the curve C) transforms $\mathscr{P}$ into a physically equivalent state.

Consider again a particular solution C of the problem. The variations in the coordinates in the transformation generated by $\varepsilon\psi$, (40), can be written as follows (taking (42) into account):

$$\begin{aligned}
\delta q_1 &= \{q_1, \varepsilon\psi\} = \varepsilon C_2 q_3, \\
\delta q_2 &= \{q_2, \varepsilon\psi\} = \varepsilon[\mathrm{d}^2 C_2/\mathrm{d}t^2 - (p_1/q_3)C_2], \\
\delta q_3 &= \{q_3, \varepsilon\psi\} = 0.
\end{aligned} \tag{47}$$

Then, by adding the increments (47) to the coordinates $q_1(t)$, $q_2(t)$ and $q_3(t)$ of the solution, we obtain an equivalent solution C'. Substituting this into the Lagrange equations (25), it may be easily checked that the coordinates of C' also satisfy the Lagrange equations. On the other hand, restricting (47) to (46), (44) we obtain that the transformation of the given point $\mathscr{P}$ induced by the generator ψ is $\delta q_2 = \delta q_3 = 0$, $\delta q_1 = \varepsilon q_3$ and coincides with that induced by the chosen secondary constraint (43).

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